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UNIFORM CONVERGENCE OF DOUBLE TRIGONOMETRIC INTEGRALS

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Abstract. We study the uniform convergence of double cosine integrals, sine-cosine integrals and double sine integrals of double general monotone functions. We extend the results of F. Móricz and of A. Debernardi concerning the uniform convergence of double sine integrals and the results of M. Dyachenko, E. Liflyand and S. Tikhonov regarding the uniform convergence of single trigonometric integrals.

1. Single trigonometric integrals. Let $\phi, \psi : \mathbb{R}_+ \to \mathbb{C}$ be Lebesgue measurable functions where $\mathbb{R}_+ := (0, \infty)$. The uniform convergence of the cosine and sine integrals

$$\int_0^\infty \phi(x) \cos tx \, dx, \quad t \in \overline{\mathbb{R}}_+, \tag{1.1}$$

$$\int_0^\infty \psi(x) \sin tx \, dx, \quad t \in \overline{\mathbb{R}}_+, \tag{1.2}$$

where $\overline{\mathbb{R}}_+ = [0, \infty)$, was studied for ϕ and ψ having certain properties. These properties often contain general monotone criteria [DLT, K1]. In this section, we always assume that $\phi(x)$ and $\psi(x)$ (or $\rho(x)$ if we do not distinguish between them) are locally of bounded variation on $\mathbb{R}_+$, written $\rho(x) \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}_+)$, and $\phi(x) \in L^1_{\mathrm{loc}}(\overline{\mathbb{R}}_+)$, $x\psi(x) \in L^1_{\mathrm{loc}}(\overline{\mathbb{R}}_+)$. We call such functions *admissible* (for (1.1) or (1.2), respectively).

Definition 1.1 ([DLT]). An admissible function ρ is called *general monotone* with majorant $\beta : \mathbb{R}_+ \to \overline{\mathbb{R}}_+$, written $\rho \in \mathrm{GMF}(\beta)$, if there exist constants $C, A > 0$ depending only on ρ and such that

$$\int_a^{2a} |d\rho(x)| \le C\beta(a) \quad \text{for } a > A.$$

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A typical example of majorant β (introduced in [LT1]) is

$$ {}_1\beta(a) = \frac{1}{a} \int_{a/c}^{ca} |\rho(x)|\, dx. $$

We recall a more general definition from [K1]:

$$ {}_2\beta(a) = \frac{1}{a} \sup_{b \ge B(a)} \int_b^{2b} |\rho(x)|\, dx $$

where $B : \mathbb{R}_+ \to \mathbb{R}_+$ increases to infinity and depends only on ρ.

For the sine integral (1.2), the following theorem was proved in [K1, Theorem 2.6], just for $\mathrm{AC}^1_{\mathrm{loc}}(\mathbb{R}_+)$ functions instead of $\mathrm{BV}_{\mathrm{loc}}(\mathbb{R}_+)$, and previously for $\psi \in \mathrm{GM}({}_1\beta)$ in [M2, Theorem 2]. See also [DLT, Theorem 3] for a similar result.

THEOREM 1.2. *Assume* $\psi \in \mathrm{GMF}({}_2\beta)$ *is an admissible function for* (1.2).

(i) *If* $f : \mathbb{R}_+ \to \mathbb{C}$ *and*

$$ x\psi(x) \to 0 \quad \text{as } x \to \infty, \tag{1.3} $$

then the sine integral (1.2) *converges uniformly in* t.

(ii) *Conversely, if* $f : \mathbb{R}_+ \to \overline{\mathbb{R}}_+$ *and* (1.2) *converges uniformly in* t, *then* (1.3) *is satisfied.*

For the cosine integral (1.1), we deduce the following theorem from [DLT, Theorem 2].

THEOREM 1.3. *Assume* $\phi \in \mathrm{GMF}({}_2\beta)$ *is an admissible function for* (1.1).

(i) *If* $f : \mathbb{R}_+ \to \mathbb{C}$ *and*

$$ x\phi(x) \to 0 \qquad \text{as } x \to \infty, \tag{1.4} $$

$$ \int_0^\infty \phi(x)\, dx \qquad \text{converges}, \tag{1.5} $$

then the cosine integral (1.1) *converges uniformly in* t.

(ii) *Conversely, if* $f : \mathbb{R}_+ \to \overline{\mathbb{R}}_+$ *and* (1.1) *converges uniformly in* t, *then* (1.4) *and* (1.5) *hold.*

Proof. For (i), in view of [DLT], it is enough to note that by (1.4),

$$ a \cdot {}_2\beta(a) = \sup_{b \ge B(a)} \int_b^{2b} |\phi(x)|\, dx \le \sup_{b \ge B(a)} |b\phi(b)| \to 0 \quad \text{as } a \to \infty. $$

For (ii), (1.5) comes from convergence at zero, while (1.4) can be obtained from (1.5) as follows:

$$\begin{aligned} a\phi(a) &= \int_a^{2a} \Big(\phi(b) - \int_a^b d\phi(x)\Big)\, db \le \int_a^{2a} \phi(x)\, dx + a \int_a^{2a} |d\phi(x)| \\ &\le \int_a^{2a} \phi(x)\, dx + C \sup_{b\ge B(a)} \int_b^{2b} \phi(x)\, dx \to 0 \quad \text{as } a \to \infty. \ \blacksquare \end{aligned}$$

We remark that the notion of general monotone functions was inspired by the concept of general monotone sequences [LT2]. This type of sequences is often used when studying the uniform convergence of trigonometric series (see, for example, [FTZ], [I], [LT3], [T1], [T2]).

2. Double trigonometric integrals. Let $f, g, h : \mathbb{R}_+^2 \to \mathbb{C}$ be Lebesgue measurable functions on $\mathbb{R}_+^2$. We study the uniform regular convergence of the double integrals

$$\int_0^\infty \int_0^\infty f(x,y) \cos ux \cos vy \, dx\, dy, \quad (u,v) \in \overline{\mathbb{R}}_+^2, \tag{2.1}$$

$$\int_0^\infty \int_0^\infty g(x,y) \sin ux \cos vy \, dx\, dy, \quad (u,v) \in \overline{\mathbb{R}}_+^2, \tag{2.2}$$

$$\int_0^\infty \int_0^\infty h(x,y) \sin ux \sin vy \, dx\, dy, \quad (u,v) \in \overline{\mathbb{R}}_+^2. \tag{2.3}$$

For background on regular convergence of double integrals, see [M3]. As in the one-dimensional case, we build upon some assumptions. Throughout, we assume that f, g, h (or r if we do not distinguish between them) are of locally bounded variation on $\mathbb{R}_+^2$, written $r(x,y) \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}_+^2)$, and moreover $f(x,y) \in L^1_{\mathrm{loc}}(\overline{\mathbb{R}}_+^2)$, $xg(x,y) \in L^1_{\mathrm{loc}}(\overline{\mathbb{R}}_+^2)$, $xyh(x,y) \in L^1_{\mathrm{loc}}(\overline{\mathbb{R}}_+^2)$. We call such functions *admissible* (for (2.1), (2.2) or (2.3), respectively). For two-variable functions of Hardy bounded variation (BV), one can consult [CA] (and also [De], [M1]).

Furthermore, we need general monotonicity (for the discrete version called double general monotone sequences see [DT1], [DT2], [DT3], [Du]). We also remark that other versions of general monotonicity were considered in [De] using functions of bounded variation, while now we consider functions locally of bounded variation, i.e. $\mathrm{BV}_{\mathrm{loc}}(\mathbb{R}_+^2) \supset \mathrm{BV}(\overline{\mathbb{R}}_+^2)$.

Definition 2.1. An admissible function $r : \mathbb{R}_+^2 \to \mathbb{C}$ is *double general monotone* with majorants $\alpha, \beta, \gamma : \mathbb{R}_+^2 \to \overline{\mathbb{R}}_+$, written $r \in \mathrm{GMF}^2(\alpha, \beta, \gamma)$, if there is a constant C depending only on r and such that

$$\int_{a_1}^{2a_1} |d_x r(x,y)| \le C\alpha(a_1,y) \quad \text{for all } a_1, y>0,$$

$$\int_{a_2}^{2a_2} |d_y r(x,y)| \le C\beta(x,a_2) \quad \text{for all } x, a_2>0,$$

$$\int_{a_1}^{2a_1}\int_{a_2}^{2a_2} |d_{xy} r(x,y)| \le C\gamma(a_1,a_2) \quad \text{for all } a_1, a_2>0.$$

A typical example of a general monotone class is $\mathrm{GMF}^2({}_1\alpha, {}_1\beta, {}_1\gamma)$ with

$$ {}_1\alpha(a_1,y) = \frac{1}{a_1}\int_{a_1/c}^{ca_1} |r(s,y)|\,ds, \quad {}_1\beta(x,a_2) = \frac{1}{a_2}\int_{a_2/c}^{ca_2} |r(x,t)|\,dt,$$

$$ {}_1\gamma(a_1,a_2) = \frac{1}{a_1a_2}\int_{a_1/c}^{ca_1}\int_{a_2/c}^{ca_2} |r(s,t)|\,ds\,dt,$$

where $c \ge 2$ is a constant depending only on r [KM].

Now we consider a more general class. Consider $\mathrm{GMF}^2({}_2\alpha, {}_2\beta, {}_2\gamma)$ with

$$ {}_2\alpha(a_1,y) = \frac{1}{a_1}\sup_{B_1(a_1)\le x\le cB_1(a_1)} \int_x^{2x} |r(s,y)|\,ds,$$

$$ {}_2\beta(x,a_2) = \frac{1}{a_2}\sup_{B_2(a_2)\le y\le cB_2(a_2)} \int_y^{2y} |r(x,t)|\,dt,$$

$$ {}_2\gamma(a_1,a_2) = \frac{1}{a_1a_2}\sup_{\max\{x,y\}\ge B_3(\max\{a_1,a_2\})} \int_x^{2x}\int_y^{2y} |r(s,t)|\,ds\,dt,$$

where $B_1, B_2, B_3 : \mathbb{R}_+ \to \mathbb{R}_+$ converge to infinity as $t \to \infty$, and $c \ge 2$, B_1, B_2, B_3 depend only on r. For a discrete version of the concept, see [K2].

Regarding the uniform regular convergence of the double sine integral (2.3), we present a $\mathrm{GMF}^2({}_1\alpha, {}_1\beta, {}_1\gamma)$ version of the main result of [KM]. We remark that an extension was proved recently in [De].

Theorem 2.2. *Assume* $h \in \mathrm{GMF}^2({}_2\alpha, {}_2\beta, {}_2\gamma)$ *is an admissible function for* (2.3).

(i) *If* $h : \mathbb{R}_+^2 \to \mathbb{C}$ *and for all* $x, y > 0$ *we have*

$$xyh(x,y) \to 0 \quad \textit{as } \max\{x,y\} \to \infty, \tag{2.4}$$

then the double sine integral (2.3) *converges in the regular sense uniformly in* $(u,v) \in \overline{\mathbb{R}}_+^2$.

(ii) *Conversely, if* $h : \mathbb{R}_+^2 \to \overline{\mathbb{R}}_+$ *and* (2.3) *regularly converges uniformly in* (u,v), *then condition* (2.4) *is satisfied.*

Furthermore, we prove analogous results for sine-cosine integrals and double cosine integrals.

THEOREM 2.3. *Assume* $g \in \mathrm{GMF}^2({}_2\alpha, {}_2\beta, {}_2\gamma)$ *is an admissible function for* (2.2).

(i) *If* $g : \mathbb{R}^2_+ \to \mathbb{C}$ *and for all* $x > 0$ *and* $0 < y < y_1$ *we have*

$$xyg(x,y) \to 0 \quad \text{and} \quad \int_y^{y_1} |xg(x,t)|dt \to 0 \quad \text{as } \max\{x,y\} \to \infty, \tag{2.5}$$

then the sine-cosine integral (2.2) *converges in the regular sense uniformly in* $(u,v) \in \overline{\mathbb{R}}^2_+$.

(ii) *Conversely, if* $g : \mathbb{R}^2_+ \to \overline{\mathbb{R}}_+$ *and* (2.2) *regularly converges uniformly in* (u,v), *then* (2.5) *holds.*

THEOREM 2.4. *Assume* $f \in \mathrm{GMF}^2({}_2\alpha, {}_2\beta, {}_2\gamma)$ *is an admissible function for* (2.1).

(i) *If* $f : \mathbb{R}^2_+ \to \mathbb{C}$ *and for all* $0 < x < x_1$ *and* $0 < y < y_1$ *we have*

$$\begin{aligned} &xyf(x,y) \to 0, \quad \int_y^{y_1} |xf(x,t)|\, dt \to 0, \quad \int_x^{x_1} |yf(s,y)|\, ds \to 0, \\ &\int_x^{x_1}\int_y^{y_1} f(s,t)\, ds\, dt \to 0 \quad \text{as } \max\{x,y\} \to \infty, \end{aligned} \tag{2.6}$$

then the double cosine integral (2.1) *converges in the regular sense uniformly in* $(u,v) \in \overline{\mathbb{R}}^2_+$.

(ii) *Conversely, if* $f : \mathbb{R}^2_+ \to \overline{\mathbb{R}}_+$ *and* (2.1) *regularly converges uniformly in* (u,v), *then the conditions in* (2.6) *are satisfied.*

COROLLARY 2.5. *The results of Theorem* 2.4 *also hold for the double integrals*

$$\int_0^\infty\int_0^\infty f(x,y)e^{iux}e^{ivy}\, dx\, dy, \quad (u,v) \in \overline{\mathbb{R}}^2_+. \tag{2.7}$$

EXAMPLE 2.6. Let

$$f(x,y) = \begin{cases} \dfrac{(-1)^r}{xy(r+1)} & \text{if } 2^r \le x,y < 2^{r+1},\ r = 0,1,\dots, \\ 0 & \text{else.} \end{cases}$$

Then f is admissible, since $f \in L^1_{\mathrm{loc}}(\overline{\mathbb{R}}^2_+) \cap \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^2_+)$ (while $f \notin L^1(\overline{\mathbb{R}}^2_+)$). It can be seen that $f \in \mathrm{GMF}^2({}_2\alpha, {}_2\beta, {}_2\gamma)$ and (2.6) holds. By Corollary 2.5, the double trigonometric integral (2.7) converges in the regular sense uniformly in (u,v).

3. Auxiliary results

LEMMA 3.1. *If* $r:\mathbb{R}_+^2\to\mathbb{C}$ *belongs to* $\mathrm{GMF}^2({}_2\alpha,{}_2\beta,{}_2\gamma)$ *and* $xyr(x,y)\to 0$ *as* $\max\{x,y\}\to\infty$, *then*

$$a_1y\int_{a_1}^{\infty}|d_xr(x,y)|\to 0\quad as\ \max\{a_1,y\}\to\infty,$$

$$xa_2\int_{a_2}^{\infty}|d_yr(x,y)|\to 0\quad as\ \max\{x,a_2\}\to\infty,$$

$$a_1a_2\int_{a_1}^{\infty}\int_{a_2}^{\infty}|d_{xy}r(x,y)|\to 0\quad as\ \max\{a_1,a_2\}\to\infty.$$

Proof. The lemma is analogous to [M2, Lemma 1], and the proof can be done in a similar way: just take $\max\{B_1(a_1),y\}>x_0$, $\max\{x,B_2(a_2)\}>x_0$ and $\max\{B_3(\max\{a_1,a_2\})\}>x_0$, respectively, where x_0 satisfies $xy|r(x,y)|<\varepsilon$ for $\max\{x,y\}>x_0$. ∎

LEMMA 3.2. *Let* $r:\mathbb{R}_+^2\to\mathbb{C}$ *belong to* $\mathrm{GMF}^2({}_2\alpha,{}_2\beta,{}_2\gamma)$.
If $\int_y^{y_1}|xr(x,t)|\,dt\to 0$ *as* $\max\{x,y\}\to\infty$, *then*

$$x\int_x^{\infty}\int_y^{y_1}|d_xr(s,t)|\,dt\to 0\quad as\ \max\{x,y\}\to\infty;$$

similarly, if $\int_x^{x_1}|yr(s,y)|ds\to 0$ *as* $\max\{x,y\}\to\infty$, *then*

$$y\int_x^{x_1}\int_y^{\infty}|d_yr(s,t)|\,ds\to 0\quad as\ \max\{x,y\}\to\infty.$$

Proof. We prove the first statement as follows:

$$\begin{aligned}
x\int_x^{\infty}\int_y^{y_1}|d_xr(s,t)|\,dt&=\int_y^{y_1}x\Big(\sum_{n=0}^{\infty}\int_{2^nx}^{2^{n+1}x}|d_xr(s,t)|\Big)\,dt\\
&\le\int_y^{y_1}\Bigg(\sum_{n=0}^{\infty}\frac{C}{2^n}\sup_{B_1(2^nx)\le a\le cB_1(2^nx)}\int_a^{2a}|r(s,t)|\,ds\Bigg)\,dt\\
&\le\sum_{n=0}^{\infty}\frac{C}{2^n}\int_{B_1(2^nx)}^{2cB_1(2^nx)}\int_y^{y_1}|r(s,t)|\,ds\,dt\\
&\le\sum_{n=0}^{\infty}\frac{C}{2^n}\cdot\frac{1}{B_1(2^nx)}\int_{B_1(2^nx)}^{2cB_1(2^nx)}\Big(\int_y^{y_1}|sr(s,t)|\,dt\Big)\,ds\to 0
\end{aligned}$$

as $\max\{x,y\}\to\infty$. The second statement can be seen in an analogous way. ∎

LEMMA 3.3. *Assume* $r : \mathbb{R}_+^2 \to \overline{\mathbb{R}}_+$ *belongs to* $\mathrm{GMF}^2({}_2\alpha, {}_2\beta, {}_2\gamma)$. *Then for all* $x, y > 0$,

$$xyr(x,y) \le C \sup_{\max\{a,b\} \ge B_3(\max\{x,y\})} \int_a^{2a}\int_b^{2b} r(s,t)\,ds\,dt + C \int_{B_1(x)}^{2cB_1(x)} \int_y^{2y} r(s,t)\,ds\,dt$$
$$+ C \int_x^{2x} \int_{B_2(y)}^{2cB_2(y)} r(s,t)\,ds\,dt + \int_x^{2x}\int_y^{2y} r(s,t)\,ds\,dt,$$

and for all $0 < x < x_1$ *and* $0 < y < y_1$,

$$\int_y^{y_1} xr(x,t)\,dt \le C \int_{B_1(a)}^{2cB_1(a)} \int_y^{y_1} r(s,t)\,ds\,dt + \int_x^{2x}\int_y^{y_1} r(s,t)\,ds\,dt,$$
$$\int_x^{x_1} yr(s,y)\,ds \le C \int_x^{x_1} \int_{B_2(y)}^{2cB_2(y)} r(s,t)\,ds\,dt + \int_x^{x_1}\int_y^{2y} r(s,t)\,ds\,dt.$$

Proof. As in [M2, Lemma 3], one can see that for all $x \le a \le 2x$ and $y \le b \le 2y$,

$$r(x,y) = \int_x^a\int_y^b d_{xy}r(s,t) - \int_x^a d_xr(s,b) - \int_y^b d_yr(a,t) + r(a,b)$$
$$\le \int_x^{2x}\int_y^{2y} |d_{xy}r(s,t)| + \int_x^{2x} |d_xr(s,b)| + \int_y^{2y} |d_yr(a,t)| + r(a,b).$$

After integrating both sides with respect to $a \in [x, 2x]$ and $b \in [y, 2y]$, we get

$$xyr(x,y) \le Cxy \cdot {}_2\gamma(x,y) + Cx \int_y^{2y} {}_2\alpha(x,b)\,db$$
$$+ Cy \int_x^{2x} {}_2\beta(a,y)\,da + \int_x^{2x}\int_y^{2y} r(a,b)\,da\,db,$$

from which we obtain the first required inequality.

Similarly, for all $x \le a \le 2x$; t and s; and $y \le b \le 2y$, we calculate

$$r(x,t) = -\int_x^a d_xr(s,t) + r(a,t) \le \int_x^{2x} |d_xr(s,t)| + r(a,t),$$
$$r(s,y) = -\int_y^b d_yr(s,t) + r(s,b) \le \int_y^{2y} |d_yr(s,t)| + r(s,b).$$

After integrating both sides of the first inequality with respect to $a \in [x, 2x]$, $t \in [y, y_1]$, and the second one over $b \in [y, 2y]$, $s \in [x, x_1]$, we obtain

$$\int_y^{y_1} xr(x,t)\,dt \le C \int_y^{y_1} \sup_{B_1(x)\le a\le cB_1(x)} \int_a^{2a} r(s,t)\,ds\,dt + \int_x^{2x}\int_y^{y_1} r(s,t)\,ds\,dt,$$

$$\int_x^{x_1} yr(s,y)\,ds \le C \int_x^{x_1} \sup_{B_2(y)\le b\le cB_2(y)} \int_b^{2b} r(s,t)\,ds\,dt + \int_x^{x_1}\int_y^{2y} r(s,t)\,ds\,dt,$$

hence the last two required inequalities are also proved. ∎

4. Proofs. Theorem 2.2 can be proved in an analogous way to [KM, Theorem 1], just with the use of Lemma 3.1 instead of [KM, Lemma 1] and Lemma 3.3 instead of [KM, Lemma 3].

Proof of Theorem 2.3. To shorten the proof, for $0 \le a_1 < b_1$ and $0 \le a_2 < b_2$, let

$$I_{uv}(g; a_1, b_1; a_2, b_2) = \int_{a_1}^{b_1}\int_{a_2}^{b_2} g(x,y)\sin ux \cos vy\,dx\,dy.$$

(i) Suppose that (2.5) is satisfied. Pick $\varepsilon > 0$. Let $x_0 = x_0(\varepsilon)$ be such that

$$|xyg(x,y)| \le \varepsilon, \qquad \int_y^{y_1} |xg(x,t)|\,dt \le \varepsilon,$$

$$x\int_x^{\infty}\int_y^{y_1} |d_x g(s,t)|\,dt \le \varepsilon, \qquad xy\int_x^{\infty} |d_x g(s,y)| \le \varepsilon, \tag{4.1}$$

$$xy\int_y^{\infty} |d_y g(x,t)| \le \varepsilon, \qquad xy\int_x^{\infty}\int_y^{\infty} |d_{xy} g(s,t)| \le \varepsilon,$$

for any $x > 0$ and $0 < y < y_1$ with $\max\{x, y\} > x_0$. From now on, we always suppose that $0 \le a_1 < b_1$, $0 \le a_2 < b_2$ and $\max\{a_1, a_2\} > x_0$. We will prove that for all $u, v > 0$ we have

$$|I_{uv}(g; a_1, b_1; a_2, b_2)| \le 20\varepsilon, \tag{4.2}$$

which is equivalent to the required uniform, regular convergence of (2.2).

From Fatou's lemma we may assume that $a_1, a_2 > 0$. For $u = 0$ and arbitrary v, (4.2) is trivial. Now let $u > 0$ and $v = 0$. If $a_1 < b_1 \le 1/u$, then by (4.1),

$$|I_{u0}(g; a_1, b_1; a_2, b_2)| \le \int_{a_1}^{b_1}\left|\int_{a_2}^{b_2} g(x,y)\,dy\right| \sin ux\,dx \le \frac{1}{b_1}\int_0^{b_1}\left|\int_{a_2}^{b_2} xg(x,y)\,dy\right| dx \le \varepsilon.$$

If $1/u \le a_1 < b_1$, then integrating by parts and using (4.1) again gives

$$\begin{aligned}(4.3)\qquad &|I_{u0}(g; a_1, b_1; a_2, b_2)|\\ &= \left|\int_{a_2}^{b_2}\left(\left[-g(x,y)\frac{\cos ux}{u}\right]_{x=a_1}^{b_1} + \int_{a_1}^{b_1}\frac{\cos ux}{u}\,d_x g(x,y)\right)dy\right|\\ &\le \int_{a_2}^{b_2}|b_1 g(b_1,y)|\,dy + \int_{a_2}^{b_2}|a_1 g(a_1,y)|\,dy + \int_{a_2}^{b_2} a_1\int_{a_1}^{\infty}|d_x g(x,y)|\,dy\\ &\le 3\varepsilon.\end{aligned}$$

For arbitrary $u, v > 0$, we distinguish four basic cases, which together guarantee (4.2). We repeatedly use (4.1) in the estimates.

Case (a): $a_1 < b_1 \le 1/u$, and $a_2 < b_2 \le 1/v$. Then

$$\begin{aligned}|I_{uv}(g; a_1, b_1; a_2, b_2)| &= \left|\int_{a_1}^{b_1}\int_{a_2}^{b_2} g(x,y)\sin ux\left(1 - 2\sin^2\frac{vy}{2}\right)dx\,dy\right|\\ &\le |I_{u0}(g; a_1, b_1; a_2, b_2)|\\ &\quad + \int_0^{1/u}\int_0^{1/v}|2uxg(x,y)|\sin\frac{vy}{2}\,dx\,dy\\ &\le \varepsilon + uv\int_0^{1/u}\int_0^{1/v}|xyg(x,y)|\,dx\,dy \le 2\varepsilon.\end{aligned}$$

Case (b): $1/u \le a_1 < b_1$ and $a_2 < b_2 \le 1/v$. Integrating by parts as in (4.3) yields

$$\begin{aligned}&|I_{uv}(g; a_1, b_1; a_2, b_2)|\\ &\quad\le |I_{u0}(g; a_1, b_1; a_2, b_2)| + \int_{a_2}^{b_2} 2\sin^2\frac{vy}{2}\left|\int_{a_1}^{b_1} g(x,y)\sin ux\,dx\right|dy\\ &\quad\le 3\varepsilon + v\int_{a_2}^{b_2}\left(|b_1 y g(b_1,y)| + |a_1 y g(a_1,y)| + a_1 y\int_{a_1}^{\infty}|d_x g(x,y)|\right)dy\\ &\quad\le 3\varepsilon + v\int_0^{1/v} 3\varepsilon\,dy \le 6\varepsilon.\end{aligned}$$

Case (c): $a_1 < b_1 \le 1/u$ and $1/v \le a_2 < b_2$. Then

$$|I_{uv}(g; a_1, b_1; a_2, b_2)| \le \int_{a_1}^{b_1}\sin ux\left|\int_{a_2}^{b_2} g(x,y)\cos vy\,dy\right|dx$$

$$\le u\left|\int_{a_1}^{b_1} x\left(\left[-g(x,y)\frac{\sin vy}{v}\right]_{y=a_2}^{b_2}+\int_{a_2}^{b_2}\frac{\sin vy}{v}d_y g(x,y)\right)dx\right|$$

$$\le u\int_0^{1/u}\left(|xb_2 g(x,b_2)|+|xa_2 g(x,a_2)|+xa_2\int_{a_2}^{\infty}|d_y g(x,y)|\right)dx\le 3\varepsilon.$$

CASE (d): $1/u \le a_1 < b_1$ and $1/v \le a_2 < b_2$. Multiple integrations by parts give

$$\begin{aligned}
&|I_{uv}(g;a_1,b_1;a_2,b_2)|\\
&=\left|\int_{a_2}^{b_2}\cos vy\left(\left[-g(x,y)\frac{\cos ux}{u}\right]_{x=a_1}^{b_1}+\int_{a_1}^{b_1}\frac{\cos ux}{u}d_x g(x,y)\right)dy\right|\\
&=\Bigg|\left[\left[-g(x,y)\frac{\cos ux}{u}\,\frac{\sin vy}{v}\right]_{x=a_1}^{b_1}\right]_{y=a_2}^{b_2}+\left[\int_{a_1}^{b_1}\frac{\cos ux}{u}\,\frac{\sin vy}{v}d_x g(x,y)\right]_{y=a_2}^{b_2}\\
&\quad+\left[\int_{a_2}^{b_2}\frac{\cos ux}{u}\,\frac{\sin vy}{v}d_y g(x,y)\right]_{x=a_1}^{b_1}-\int_{a_1}^{b_1}\int_{a_2}^{b_2}\frac{\cos ux}{u}\,\frac{\sin vy}{v}d_{xy} g(x,y)\Bigg|\\
&\le |b_1b_2\,g(b_1,b_2)|+|a_1b_2\,g(a_1,b_2)|+|b_1a_2\,g(b_1,a_2)|+|a_1a_2\,g(a_1,a_2)|\\
&\quad+a_1b_2\int_{a_1}^{\infty}|d_x g(x,b_2)|+a_1a_2\int_{a_1}^{\infty}|d_x g(x,a_2)|+b_1a_2\int_{a_2}^{\infty}|d_y g(b_1,y)|\\
&\quad+a_1a_2\int_{a_2}^{\infty}|d_y g(a_1,y)|+a_1a_2\int_{a_1}^{\infty}\int_{a_2}^{\infty}|d_{xy} g(x,y)|\le 9\varepsilon.
\end{aligned}$$

(ii) For arbitrary $x > 0$ and $0 < y < y_1$, set $u = \frac{\pi}{4cx}$. Clearly, for all $x \le s \le 2cx$, we have $\sin us \ge \sin\frac{\pi}{4c}$, hence

$$|I_{u0}(g;x,cx;y,y_1)| = \int_x^{2cx}\int_y^{y_1} g(s,t)\sin us\,ds\,dt \ge \sin\frac{\pi}{4c}\int_x^{2cx}\int_y^{y_1} g(s,t)\,ds\,dt.$$

Then from the uniform, regular convergence of (2.2), and Lemma 3.3, we obtain (2.5). ∎

Proof of Theorem 2.4. For $0 \le a_1 < b_1$ and $0 \le a_2 < b_2$, let

$$I_{uv}(f;a_1,b_1;a_2,b_2) = \int_{a_1}^{b_1}\int_{a_2}^{b_2} f(x,y)\cos ux\cos vy\,dx\,dy.$$

(i) Suppose that (2.6) is satisfied. Pick $\varepsilon > 0$. Choose $x_0 = x_0(\varepsilon)$ so that

$$\begin{aligned}
&|xyf(x,y)| \le \varepsilon, \quad \int_y^{y_1} |xf(x,t)|\,dt \le \varepsilon, \quad \int_x^{x_1} |yf(s,y)|\,ds \le \varepsilon,\\
&xy\int_x^{\infty} |d_x f(s,y)| \le \varepsilon, \quad xy\int_y^{\infty} |d_y f(x,t)| \le \varepsilon,\\
&x\int_x^{\infty}\int_y^{y_1} |d_x f(s,t)|\,dt \le \varepsilon, \quad y\int_x^{x_1}\int_y^{\infty} |d_y f(s,t)|\,ds \le \varepsilon,\\
&xy\int_x^{\infty}\int_y^{\infty} |d_{xy} f(s,t)| \le \varepsilon, \quad \left|\int_x^{x_1}\int_y^{y_1} f(s,t)\,ds\,dt\right| \le \varepsilon,
\end{aligned} \tag{4.4}$$

for all $0 < x < x_1$ and $0 < y < y_1$ with $\max\{x,y\} > x_0$. From now on, we always assume that $0 \le a_1 < b_1$, $0 \le a_2 < b_2$ and $\max\{a_1,a_2\} > x_0$. We will prove that for all u, v,

$$|I_{uv}(f;a_1,b_1;a_2,b_2)| \le 25\varepsilon. \tag{4.5}$$

From Fatou's lemma we may suppose that $a_1, a_2 > 0$. If $u = v = 0$, then the last inequality in (4.4) immediately implies (4.5). In the following, (4.4) is used several times. Let $u > 0$ and $v = 0$. If $a_1 \le b_1 \le 1/u$, then

$$\begin{aligned}
|I_{u0}(f;a_1,b_1;a_2,b_2)| &\le \left|\int_{a_1}^{b_1}\int_{a_2}^{b_2} f(x,y)\left(1 - 2\sin^2\frac{ux}{2}\right)dx\,dy\right|\\
&\le \left|\int_{a_1}^{b_1}\int_{a_2}^{b_2} f(x,y)\,dx\,dy\right| + \frac{1}{b_1}\int_0^{b_1}\left|\int_{a_2}^{b_2} xf(x,y)\,dy\right|dx \le 2\varepsilon.
\end{aligned}$$

If $1/u \le a_1 \le b_1$, then integrating by parts and a similar argument to (4.3) give

$$\begin{aligned}
&|I_{u0}(f;a_1,b_1;a_2,b_2)|\\
&\qquad = \left|\int_{a_2}^{b_2}\left(\left[f(x,y)\frac{\sin ux}{u}\right]_{x=a_1}^{b_1} + \int_{a_1}^{b_1}\frac{\sin ux}{u}d_x f(x,y)\right)dy\right| \le 3\varepsilon.
\end{aligned}$$

Analogously, for $u = 0$ and $v > 0$ we can obtain

$$|I_{0v}(f;a_1,b_1;a_2,b_2)| \le 2\varepsilon + 3\varepsilon = 5\varepsilon.$$

For any $u, v > 0$, as in the proof of Theorem 2.3, we distinguish four basic cases, which together guarantee (4.5).

CASE (a): $a_1 < b_1 \le 1/u$ and $a_2 < b_2 \le 1/v$. Then

$$\begin{aligned}\|I_{uv}(f;a_1,b_1;a_2,b_2)| &= \left|\int_{a_1}^{b_1}\int_{a_2}^{b_2} f(x,y)\left(1-2\sin^2\frac{ux}{2}\right)\left(1-2\sin^2\frac{vy}{2}\right)dx\,dy\right| \\ &\le \left|\int_{a_1}^{b_1}\int_{a_2}^{b_2} f(x,y)\,dx\,dy\right| + u\int_0^{1/u}\left|\int_{a_2}^{b_2} xf(x,y)\,dy\right|dx \\ &\quad + v\int_0^{1/v}\left|\int_{a_1}^{b_1} yf(x,y)\,dx\right|dy + uv\int_0^{1/u}\int_0^{1/v} xy\,|f(x,y)|\,dx\,dy \le 4\varepsilon.\end{aligned}$$

CASE (b): $1/u \le a_1 < b_1$ and $a_2 < b_2 \le 1/v$. Analogously to Case (b) of the proof of Theorem 2.3,

$$\begin{aligned}&|I_{uv}(f;a_1,b_1;a_2,b_2)| \\ &\quad \le |I_{u0}(f;a_1,b_1;a_2,b_2)| + \int_{a_2}^{b_2} 2\sin^2\frac{vy}{2}\left|\int_{a_1}^{b_1} f(x,y)\cos ux\,dx\right|dy \le 6\varepsilon.\end{aligned}$$

CASE (c): $a_1 < b_1 \le 1/u$ and $1/v \le a_2 < b_2$. This is the symmetric counterpart of Case (b), hence

$$|I_{uv}(f;a_1,b_1;a_2,b_2)| \le 6\varepsilon.$$

CASE (d): $1/u \le a_1 < b_1$ and $1/v \le a_2 < b_2$. Analogously to Case (d) of the proof of Theorem 2.3, multiple integrations by parts yields

$$|I_{uv}(f;a_1,b_1;a_2,b_2)| \le 9\varepsilon.$$

(ii) From the regular convergence of (2.1) at $(0,0)$ we deduce that

$$I_{00}(f;x,x_1;y,y_1) = \int_x^{x_1}\int_y^{y_1} f(s,t)\,ds\,dt \to 0 \quad \text{as } \max\{x,y\} \to \infty,$$

and from Lemma 3.3 we obtain (2.6). ∎

Proof of Corollary. (i) Obviously,

$$\begin{aligned}\iint f(x,y)e^{iux}e^{ivy} &= \iint f(x,y)\cos ux\cos vy + i\iint f(x,y)\sin ux\cos vy \\ &\quad + i\iint f(x,y)\cos ux\sin vy - \iint f(x,y)\sin ux\sin vy,\end{aligned}$$

and we can apply Theorems 2.2–2.4.

(ii) We just need to use the regular convergence of (2.7) at $(0,0)$ and Lemma 3.3 as in the proof of Theorem 2.4. ∎

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